BOUNDARY EFFECTS AND INTERPLAY BETWEEN SPONTANEOUS AND ANOMALOUS BREAKING OF PARITY IN ODD DIMENSIONS

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Eta-function regularization of odd-dimensional fermionic determinants in the presence of nonvanishing background gauge field-strengths at euclidean space–time infinity yields both anomalous and spontaneous violation of parity. The competitive contributions of the latter to the magnitude of induced currents and charges in a static or constant uniform background are calculated.

1. Some time ago Polyakov [1] proposed a general formula for a gauge-invariant regularization of D-dimensional euclidean fermionic determinants (one-loop effective actions) for arbitrary Dirac operators, whose spectrum is not symmetric around $\lambda = 0$, in terms of the spectral asymmetry measuring η-invariant of Atiyah, Patodi and Singer [2] (see also ref. [3]):

$$N_f \ln \det[-i\nabla\!\!\!/(A,\varphi)] = \tfrac{1}{2} N_f \ln \det[\nabla\!\!\!/^2(A,\varphi)] - \tfrac{1}{2} i\pi N_f \eta_{\nabla\!\!\!/}[A,\varphi] \,, \tag{1}$$

$$\ln \det[\nabla\!\!\!/^2(A,\varphi)] = -\int_0^\infty d\tau\, \tau^{-1} \operatorname{Tr}_R \{\exp[-\tau \nabla\!\!\!/^2(A,\varphi)]\} = -(d/ds)\, \zeta_{\nabla\!\!\!/^2(A,\varphi)}(s)|_{s=0} \,, \tag{2}$$

$$\eta_{\nabla\!\!\!/}[A,\varphi] = \int_{-\infty}^\infty d\lambda\, \operatorname{sign}(\lambda) \operatorname{Tr}_R[\mathcal{P}_{\nabla\!\!\!/(A,\varphi)}(\lambda)] = \int_0^\infty d\tau (\pi\tau)^{-1/2} \operatorname{Tr}_R\{\nabla\!\!\!/(A,\varphi) \exp[-\tau\nabla\!\!\!/^2(A,\varphi)]\} \,. \tag{3}$$

Here the following notations are used: $\nabla\!\!\!/(A,\varphi) = \gamma_\mu[\partial_\mu + iA_\mu(x)] + \varphi(x)$, $\{\gamma_\mu, \gamma_\nu\} = -2\delta_{\mu\nu}$, $\gamma_\mu^* = -\gamma_\mu$, $A_\mu(x) = T^a A_\mu^a(x)$, $\{T^a\}$ ($a = 0, 1, ..., n^2-1$) are the hermitian generators of $G = U(n)$, $\varphi(x) = \varphi_1(x) + \gamma^5\varphi_2(x)$ (for odd D $\varphi_2 \equiv 0$); $x \in R^D$; N_f denotes the number of fermion flavors (i.e. $\psi_j(x), j = 1, ..., N_f$); $\mathcal{P}_{\nabla\!\!\!/(A,\varphi)}(\lambda; x, x')$ is the kernel of the spectral density of $\nabla\!\!\!/(A,\varphi)$, ($\nabla\!\!\!/(A,\varphi) = \int \lambda \mathcal{P}_{\nabla\!\!\!/(A,\varphi)}(\lambda)\, d\lambda$) and $\operatorname{Tr}_R[\ ]$ indicates volume (infrared) regularized operator trace. In (1)–(3) standard boundary conditions on $A_\mu(x)$ and on $\varphi(x)$ are assumed:

$$A_\mu(x) = -ig^{-1}(\hat{x})(\partial_\mu g)(\hat{x}) + O(|x|^{-1-\epsilon}) \,, \qquad \varphi(x) = O(|x|^{-1-\epsilon}) \qquad \text{for } |x| \to \infty \,,$$

$$\hat{x} = (x/|x|) \in S^{D-1} \,, \quad g: S_\infty^{D-1} \to U(n) \,. \tag{4}$$

In case of (4) $\operatorname{Tr}_R[\ ]$ may be defined by subtraction of the corresponding operator at $A_\mu = \varphi = 0$ [‡1]. Eq. (2) rep-

[‡1] More generally, regularized operator traces are defined as $\operatorname{Tr}_R[\ ] = \operatorname{Tr}[\chi_V(\ )]$, where χ_V denotes the multiplication operator by the characteristic function of a compact subset $V \subset R^D$. At the end of the computations for the induced current $J_\mu^a(x)$ (6) we shall take $V \to R^D$.

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resents the usual ζ-function regularization [4] of the determinant of the nonnegative operator $\not{\nabla}^2(A,\varphi)$. The η-invariant of $\not{\nabla}(A,\varphi)$ (3) is odd under parity transformation in any odd D:

$$\psi^{\mathrm{P}}(x)=-\mathrm{i}\gamma_1\psi(x^{\mathrm{P}}),\quad A_\mu^{\mathrm{P}}(x)=(A_0,-A_1,\dots,A_{D-1})(x^{\mathrm{P}}),\quad \varphi_1^{\mathrm{P}}(x)=-\varphi_1(x^{\mathrm{P}}),$$

$$\not{\nabla}(A^{\mathrm{P}},\varphi^{\mathrm{P}})(x,x')=\gamma_1\not{\nabla}(A,\varphi)(x^{\mathrm{P}},x'^{\mathrm{P}})\gamma_1,\quad \mathcal{P}_{\not{\nabla}(A^{\mathrm{P}},\varphi^{\mathrm{P}})}(\lambda;x,x')=-\gamma_1\mathcal{P}_{\not{\nabla}(A,\varphi)}(-\lambda;x^{\mathrm{P}},x'^{\mathrm{P}})\gamma_1,$$

$$x^{\mathrm{P}}\equiv(x^0,-x^1,x^2,\dots,x^{D-1}),\tag{5}$$

and, therefore, its appearance in (1) gives rise to a parity-violating anomaly (PVA) in odd space–time dimensions [5–7]. The scalar Higgs field $\varphi(x)$ does not contribute to the PVA in odd D, so we take henceforth $\varphi(x)=0$. Also, recall that a fermion mass term explicitly breaks parity (5):

$$m(\bar{\psi}^{\mathrm{P}}\psi^{\mathrm{P}})(x)=-m(\bar{\psi}\psi)(x^{\mathrm{P}}).\tag{5'}$$

The aim of the present note is to study the induced fermion current in the presence of a background $A_\mu(x)$

$$J_\mu^a(x)=\langle\bar{\psi}_{\mathrm{j}}(x)\,T^a(-\mathrm{i}\gamma_\mu)\,\psi_{\mathrm{j}}(x)\rangle=-\mathrm{i}[\delta/\delta A_\mu^a(x)]\,N_{\mathrm{f}}\ln\det[-\mathrm{i}\not{\nabla}(A)],\tag{6}$$

where $A_\mu(x)$ obeys boundary conditions more general than (4) and an improved version of (1) (see (10), (12) below) will be invoked so as to properly incorporate all nonperturbative parity-breaking effects (anomalous and spontaneous) ‡2. This will lead to some substantial amendments of results obtained in refs. [5,7].

2. As a consequence of the Atiyah–Patodi–Singer index theorem [2], in the case of boundary conditions (4) (which allow compactification of R^D to S^D) one has

$$\eta_{\not{\nabla}}[A]=(-1)^{(D+1)/2}\,2W_{\mathrm{ChS}}^{(D)}[A]+B[A],\tag{7}$$

where $W_{\mathrm{ChS}}^{(D)}[A]$ denotes the well-known odd-dimensional (and parity-odd) Chern–Simons term (e.g. ref. [8]) and $B[A]$ is twice the index of an appropriate $(D+1)$(=even)-dimensional Dirac operator.

Formula (1) together with (7) was further analyzed in ref. [9]. In order to account for the renormalization ambiguity, a finite counterterm $S_{\mathrm{c.t.}}[A]$ should in general be added to the right-hand side of (1) obeying the following properties:

(a) $S_{\mathrm{c.t.}}[A]$ must be a local gauge-invariant functional of $A_\mu(x)$ of dimension D.

(b) $S_{\mathrm{c.t.}}[A]$ must be chosen in such a way as to eventually cancel the PVA coming from $\eta_{\not{\nabla}}[A]$ in (1).

Accounting for the well-known properties of $W_{\mathrm{ChS}}^{(D)}[A]$ and of the topological charge $N_D[g]$ of gauge transformations $g(x)\in\mathrm{U}(n)$:

$$W_{\mathrm{ChS}}^{(D)}[A^g]=W_{\mathrm{ChS}}^{(D)}[A]+N_{\mathrm{D}}[g],$$

$$N_D[g]=-(\mathrm{i}/2\pi)^{(D+1)/2}(\tfrac{1}{2}(D-1))!(D!)^{-1}\epsilon_{\mu_1\dots\mu_D}\int\mathrm{d}^Dx\ \mathrm{tr}[(g^{-1}\partial_{\mu_1}g)\dots(g^{-1}\partial_{\mu_D}g)],\tag{8a}$$

$$N_D[g]\in\mathrm{Z}\quad\text{if }\pi_D(\mathrm{U}(n))=\mathrm{Z},\quad\text{i.e. for odd }D<2n,$$

$$=0\quad\text{if }\pi_D(\mathrm{U}(n))\neq\mathrm{Z},\quad\text{i.e. for odd }D>2n,\tag{8b}$$

where $A_\mu^g(x)=g^{-1}(x)[A_\mu(x)-\mathrm{i}\partial_\mu]\,g(x)$, $\pi_D(\mathrm{G})$ denotes the Dth homotopy group of G and Z is the group of in-

‡2 Let us recall the basic difference between spontaneous and anomalous symmetry breaking. Spontaneous breaking takes place due to the degeneracy of the ground state for certain subspaces of the coupling parameter (or of the external background field-) space and the symmetry may be restored for other subspaces of the latter. On the contrary, anomalous breaking is an unavoidable violation of the classical symmetry for any (nonzero) values of the coupling parameters (or of the external fields).

tegers, one easily finds the following two alternatives:

(i) If either $\pi_D(\mathrm{U}(n)) \neq \mathrm{Z}$ (i.e. $D > 2n$) or if $\pi_D(\mathrm{U}(n)) = \mathrm{Z}$ (i.e. $D < 2n$) and N_f = even simultaneously:

$$N_f S_{\text{c.t.}}[A] = \mathrm{i}\pi(-1)^{(D-1)/2} N_f W_{\text{ChS}}^{(D)}[A] , \tag{9}$$

and then formula (1) together with (7)–(9) reads

$$N_f \ln \det[-\mathrm{i}\nabla\!\!\!/(A)] = \tfrac{1}{2} N_f \ln \det[\nabla\!\!\!/^2(A)] - \tfrac{1}{2}\mathrm{i}\pi N_f B[A] . \tag{10}$$

$\frac{1}{2}\pi N_f B[A] = 0 \pmod{2\pi}$ under the above conditions, and thus the PVA in (10) is eliminated.

(ii) If $\pi_D(\mathrm{U}(n)) = \mathrm{Z}$ (i.e. $D < 2n$) and N_f = odd simultaneously, the choice (9) is unacceptable since it breaks gauge invariance except for the U(1) subgroup, cf. (8). Therefore, in this case

$$S_{\text{c.t.}}[A] = \mathrm{i}\pi(-1)^{(D-1)/2} W_{\text{ChS}}^{(D)}[A^{a=0}] , \tag{11}$$

$$N_f \ln \det[-\mathrm{i}\nabla\!\!\!/(A)] = \tfrac{1}{2} N_f \ln \det[\nabla\!\!\!/^2(A)] - \tfrac{1}{2}\mathrm{i}\pi N_f B[\hat{A}] + \mathrm{i}\pi(-1)^{(D-1)/2} N_f W_{\text{ChS}}^{(D)}[\hat{A}] , \tag{12}$$

$A_\mu^{a=0}(x)$, $\hat{A}_\mu(x)$ being the abelian and the SU(n) parts of $A_\mu(x)$, respectively. Hence a PVA is inavoidable in (12).

3. Let us now consider the η-function regularization of the odd-D fermionic determinants (10), (12) in the case of nonvanishing boundary conditions more general than (4) for the background field strength $F_{\mu\nu}(A) = \partial_\mu A_\nu - \partial_\nu A_\mu + \mathrm{i}[A_\mu, A_\nu]$

$$F_{\mu\nu}(A) \xrightarrow[|x|\to\infty]{} F_{\mu\nu}^\infty \neq 0 \qquad \text{(in certain or in all directions in } \mathrm{R}^D) . \tag{13}$$

In particular, (13) means that $A_\mu(x)$ is just an external background (not quantized) field. (13) will now lead to nontrivial boundary effects in (10), (12), (6). Namely, $B[A]$ in (7) can no longer be identified as twice the index of a certain $(D + 1)$-dimensional Dirac operator (boundary conditions (13) do not allow compactification of R^D anymore). Under (13) $B[A]$ becomes a nontrivial smoothly varying nonlocal functional of $A_\mu(x)$. The latter produces an additional parity breaking not present in (10), (12) under standard boundary conditions (4) and which cannot be compensated by a local $S_{\text{c.t.}}[A]$.

From the heat kernel representation of $\eta_{\nabla\!\!\!/}[A]$ (3) and from (7) one easily finds

$$[\delta/\delta A_\mu^a(x)]\, B[A] = 2\mathrm{i}\, \mathrm{tr}[T^a \gamma_\mu \mathcal{P}_{\nabla\!\!\!/(A)}(0;x,x)] , \tag{14}$$

where the formula

$$\exp[-\tau\nabla\!\!\!/^2(A)](x,x) \approx (\pi/\tau)^{1/2} \mathcal{P}_{\nabla\!\!\!/(A)}(0;x,x) \qquad \text{for } \tau \to \infty \tag{15}$$

was used. Unlike (4) where

$$\mathcal{P}_{\nabla\!\!\!/(A)}(0;x,x') = \delta(0)\, \Pi_0^{\nabla\!\!\!/(A)}(x,x') = \infty , \qquad \text{if } \lambda = 0 \text{ is a discrete eigenvalue} ,$$

$$= 0 , \qquad \text{otherwise} , \tag{16a}$$

(cf. e.g. ref. [9]) with $\Pi_0^{\nabla\!\!\!/(A)}(x,x')$ being the kernel of the zero mode projector, in the case of (13) we have in general

$$\mathcal{P}_{\nabla\!\!\!/(A)}(0;x,x) \neq 0 \ (<\infty) . \tag{16b}$$

Note that $\lambda = 0$ in (16b) belongs to the absolutely continuous part of the spectrum of $\nabla\!\!\!/(A)$ (not a zero eigenvalue). Let us emphasize that (16b), accounting for (16a), yields in odd D spontaneous (not anomalous) breaking of parity in the presence of the background $A_\mu(x)$. This can also be inferred from

$$\langle\bar{\psi}\psi\rangle(x) = -\lim_{m\to 0} \mathrm{tr}[(m + \mathrm{i}\not{\nabla}(A))^{-1}(x,x)]$$

$$= \lim_{m\to 0} \mathrm{i} \int_0^\infty \mathrm{d}\tau \exp(-\tau m^2)\, \mathrm{tr}[\{\not{\nabla}(A) \exp[-\tau\not{\nabla}^2(A)]\}(x,x)]$$

$$- \lim_{m\to 0} m^{-1} \int_0^\infty \mathrm{d}\alpha \exp(-\alpha)\, \mathrm{tr}[\exp[-(\alpha/m^2)\, \not{\nabla}^2(A)]]$$

$$= \mathrm{i} \int_{-\infty}^{\infty} \mathrm{d}\lambda P(1/\lambda)\, \mathrm{tr}[\mathcal{P}_{\not{\nabla}(A)}(\lambda; x,x)] - \pi\, \mathrm{sign}(m)\, \mathrm{tr}[\mathcal{P}_{\not{\nabla}(A)}(0;x,x)] , \tag{17}$$

where once again (15) was used. According to (5), (5′), the second term in the last equality (17) represents spontaneous breakdown (while the first one is parity-covariant). Eq. (17) completely parallels the criterion for spontaneous chiral symmetry breaking in the presence of a background $A_\mu(x)$ in even D [10].

In terms of the induced current (6), accounting for (10), (12), (14), we get

$$J^a_\mu(x) = N_\mathrm{f} \int_0^\infty \mathrm{d}\tau\, \mathrm{tr}[T^a\gamma_\mu\{\not{\nabla}(A) \exp[-\tau\not{\nabla}^2(A)]\}(x,x)] - \mathrm{i}\pi N_\mathrm{f}\, \mathrm{tr}[T^a\gamma_\mu\mathcal{P}_{\not{\nabla}(A)}(0;x,x)]$$

$$+ 0 , \quad \text{if conditions (i) hold ,}$$

$$+ \tfrac{1}{2}N_\mathrm{f}(-1)^{(D-1)/2}[(\tfrac{1}{2}(D-1))!(4\pi)^{(D-1)/2}]^{-1}\epsilon_{\mu\mu_1\ldots\mu_{D-1}}\, \mathrm{tr}[T^a\hat{F}_{\mu_1\mu_2}\ldots\hat{F}_{\mu_{D-2}\mu_{D-1}}] ,$$

$$\text{if conditions (ii) hold .} \tag{18}$$

Here $\hat{F}_{\mu\nu} = F_{\mu\nu}(\hat{A})$ denotes the SU(n) part of $F_{\mu\nu}(A)$. Let us point out that (18) could equally well be derived by means of the Pauli–Villars regularization

$$\langle\bar{\psi}(x)\, T^a(-\mathrm{i}\gamma_\mu)\, \psi(x)\rangle = \lim_{m\to 0}\left(\lim_{|M|\to\infty} \mathrm{tr}\, [T^a\gamma_\mu\{[\not{\nabla}(A) - \mathrm{i}m]^{-1}(x,x) - [\not{\nabla}(A) - \mathrm{i}M]^{-1}(x,x)\}]\right)$$

$$+ \mathrm{i}[\delta/\delta A^a_\mu(x)]\, S_{\mathrm{c.t.}}[A] .$$

4. There exist two important particular cases of nontrivial boundary conditions (13) when $\mathcal{P}_{\not{\nabla}(A)}(0;x,x)$ can be explicitly found.

First, consider the case of static $A_\mu(x)$ with zero electric field $F_{0k} = 0$ ($k = 1, ..., D-1$); then the gauge $A_0 = 0$ may be imposed. For static fields we have

$$\mathcal{P}_{\not{\nabla}_D}(0;x,x) = (2\pi)^{-1}\Pi_0^{\not{\nabla}_{D-1}}(x,x) , \tag{19}$$

$$\not{\nabla}_D \equiv \not{\nabla}(A) = \gamma_0\partial_0 + \not{\nabla}_{D-1} , \qquad \not{\nabla}_{D-1} = \gamma_k[\partial_k + \mathrm{i}A_k(x)] ,$$

$$A_k(x) = -\mathrm{i}h(\hat{x})(\partial_k h)(\hat{x}) + \mathrm{O}(|x|^{-1-\epsilon}) \quad \text{for } |x|\to\infty , \quad h: S_\infty^{D-2} \to \mathrm{U}(n) , \tag{20}$$

and $\Pi_0^{\not{\nabla}_{D-1}}(x,x')$ denotes the kernel of the zero mode projector of the $(D-1)$(=even)-dimensional Dirac operator $\not{\nabla}_{D-1}$. Remembering that $\gamma_0(\text{odd } D) = \mathrm{i}(-1)^{(D+1)/2}\gamma^5$ (even $D-1$), $\gamma_k(\text{odd } D) = \gamma_k(\text{even } D-1)$, and substituting (19) into (18) an entirely parity-odd result for $J_0^a(x)$ follows (we confine ourselves with the singlet current, i.e. for $a = 0$):

$$N_\mathrm{f}^{-1} J_0^{a=0}(x) = \begin{cases} (-1)^{(D+1)/2}\,\tfrac{1}{2}\,\mathrm{index}(\not{\nabla}_{D-1};x)\,, & \text{if conditions (i) hold}\,,\\ \tfrac{1}{2}(-1)^{(D+1)/2} C_{(D-1)/2}(F^{a=0};x) + (-1)^{(D-1)/2}\,\tfrac{1}{2}\left[C_{(D-1)/2}(F;x) - \mathrm{index}(\not{\nabla}_{D-1};x)\right], & \text{if conditions (ii) hold}\,. \end{cases} \tag{21}$$

Here index $(\not{\nabla}_{D-1};x) = \mathrm{tr}[\gamma^5 \Pi_0^{\not{\nabla}_{D-1}}(x,x)]$ means the index density of $\not{\nabla}_{D-1}$, $F_{\mu\nu}^{a=0}$ is the abelian part of $F_{\mu\nu}(A)$, and $C_{(D-1)/2}(F;x)$ denotes the well-known Chern characteristic class ("instanton" number of $A_k(x)$) in $D-1$ (=even) dimensions (e.g. ref. [8]). Also, note that the last line in the second equality (21) is just (up to a sign) $\partial_k\langle\bar{\psi}(x)(-\mathrm{i}\gamma_k)\,\gamma^5\psi(x)\rangle$, the divergence of the $(D-1)$-dimensional induced axial current. Thus, accounting for the standard index theorem (e.g. ref. [8]):

$$\int \mathrm{d}^{D-1}x\ \mathrm{index}(\not{\nabla}_{D-1};x) = \int \mathrm{d}^{D-1}x\ C_{(D-1)/2}(F;x) = N_{D-2}[h]$$

(h is the same as in (20)), (21) yields the following result for the induced charge:

$$Q_\mathrm{ind} = \int \mathrm{d}^{D-1}x\ J_0^{a=0}(x) = \begin{cases} N_\mathrm{f}(-1)^{(D+1)/2}\,\tfrac{1}{2}N_{D-2}[h] & \text{(for conditions (i))}\,,\\ 0 \quad (\mathrm{G} = \mathrm{SU}(n)) & \text{(for conditions (ii))}\,. \end{cases} \tag{22}$$

From (22) and (8b) we conclude that (recall D = odd):

(a) For G = U(n) Q_ind is fractional (half-integer) only when $D = 2n+1$ and N_f = odd.
(b) For G = U(n) Q_ind is a (nonzero) integer when $3 \leqslant D \leqslant 2n+1$ and N_f = even.
(c) For G = SU(n) $Q_\mathrm{ind} = 0$ identically when either $D \neq 2n+1$ and N_f = odd or $D \geqslant 2n+3$ and N_f = even.

Next, consider another particular case of (13) in $D = 3$:

$F_{\mu\nu}(A(x)) \to F_{\mu\nu}^\infty$ = constant, uniformly sufficiently fast at $|x| \to \infty$. For the spectral density at $\lambda = 0$ one has

$$\mathcal{P}_{\not{\nabla}(A)}(0;x,x) = [U_\pm \mathcal{P}_{\not{\nabla}(A^\mathrm{as})}(0)\,U_\pm^*](x,x)\,,\qquad \mathcal{P}_{\not{\nabla}(A^\mathrm{as})}(0;x,x') = (16\pi^2)^{-1}\epsilon_{\mu\nu\lambda}(-\mathrm{i}\gamma_\mu)\,F_{\nu\lambda}^\infty \tag{23}$$

(cf. e.g. ref. [11]), where $F_{\mu\nu}(A^\mathrm{as}) \equiv F_{\mu\nu}^\infty$ and $U_\pm$ denote the wave operators (in the sense of scattering theory) for $H \equiv \not{\nabla}^2(A)$ and $H_0 \equiv \not{\nabla}^2(A^\mathrm{as})$ (H and H_0 being the total and "free" quantum mechanical hamiltonians, respectively). Substituting (23) into (18) we obtain:

$$N_\mathrm{f}^{-1}J_\mu^a(x) = n(8\pi)^{-1}\epsilon_{\kappa\lambda\nu}F_{\lambda\nu}^{\infty,b}\,w_{\mu\kappa}^{ab}(x) + \text{parity-normal terms}\,, \tag{24a}$$

either for G = U(1) or for G = SU(n), $n \geqslant 2$, and N_f = even;

$$N_\mathrm{f}^{-1}J_\mu^a(x) = N_\mathrm{f}^{-1}J_\mu^a(x)(\text{eq. (24a)}) - n(8\pi)^{-1}\epsilon_{\mu\nu\lambda}F_{\nu\lambda}^a(A(x))\,, \tag{24b}$$

for G = SU(n), $n \geqslant 2$, and N_f = odd;

$$w_{\mu\kappa}^{ab}(x) \equiv -(2n)^{-1}\,\mathrm{tr}\left[T^a\gamma_\mu\left(\int \mathrm{d}^3y\ U_\pm(x,y)\right)T^b\gamma_\kappa\left(\int \mathrm{d}^3y\ U_\pm^*(y,x)\right)\right]$$
$$= \delta_{\mu\kappa}\delta^{ab} + \text{nonlocal functional of } (A_\mu(x) - A_\mu^\mathrm{as}(x))\,.$$

(24a) differs from the result found for G = U(1) in refs. [5,7] where the local form $n(8\pi)^{-1}\epsilon_{\mu\nu\lambda}F_{\nu\lambda}(A(x))$ for the corresponding parity-breaking term (i.e. a wouldbe PVA) was claimed. The present analysis shows that the parity-odd term in (24a) is entirely due to the spontaneous breakdown of parity through (23), (18), i.e. it does not represent a PVA. In particular, it vanishes in the case of the standard boundary conditions (4) and, therefore, it cannot be a local functional of $A_\mu(x)$.

To recapitulate, we have shown that a consistent determination of induced fermion currents and charges in the presence of background fields in odd D (formulae (18), (21), (22), (24)) must be based on formulae (10), (12) for the fermionic determinant which correctly includes both the effect of the spectral asymmetry of the Dirac operator, i.e. the parity-violating anomaly and the additional spontaneous parity breakdown, as well as the possibility of adding appropriate finite local counterterms eventually cancelling the anomaly.

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